Two-point resistance of a resistor network embedded on a globe

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We consider the problem of two-point resistance in an $(m-1)\times n$ resistor network embedded on a globe, a geometry topologically equivalent to an $m\times n$ cobweb with its boundary collapsed into one single point. We deduce a concise formula for the resistance between any two nodes on the globe using a method of direct summation pioneered by one of us [Z.-Z. Tan, L. Zhou, and J. H. Yang, J. Phys. A: Math. Theor. **46**, 195202 (2013)]. This method is contrasted with the Laplacian matrix approach formulated also by one of us [F. Y. Wu, J. Phys. A: Math. Gen. **37**, 6653 (2004)], which is difficult to apply to the geometry of a globe. Our analysis gives the result in the form of a single summation.

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I. INTRODUCTION

A classic problem in electric circuit theory first studied by Kirchhoff [1] more than 160 years ago is the computation of resistances in resistor networks. Kirchhoff formulated the problem in terms of the Laplacian matrix of the network and also noted that the Laplacian also generates spanning trees. For the explicit computation of two-point resistances, Venezian [2] considered the resistance between two arbitrary nodes using the method of superposition. Cserti [3] evaluated the two-point resistance using the lattice Green's function. Their studies are confined to regular lattices of infinite size.

One of the present authors [4] formulated a different approach and derived an expression for the two-point resistance in arbitrary finite and infinite lattices in terms of the eigenvalues and eigenvectors of the Laplacian matrix. The Laplacian analysis has also been extended to impedance networks after a slight modification of the formulation of [5]. We refer to these methods as the Laplacian approach.

Applications of the Laplacian approach require a complete knowledge of the eigenvalues and eigenvectors of the Laplacian straightforward to obtain for regular lattices. However, the actual computation depends crucially on the geometry of the network and, for nonregular lattices such as a cobweb or a globe, it can be difficult to solve the eigenvalue problem. Alternate methods of evaluation are needed.

The cobweb is a two-dimensional rectangular network with a periodic boundary condition imposed in one spatial direction, together with the insertion of an additional node connected to every node on one of the two boundaries. An example of the cobweb is shown in the left panel of Fig. 1. Tan, Zhou, and Yang [6] proposed a conjecture on the resistance between two nodes on the cobweb. It is difficult to adopt the Laplacian approach directly to the problem due to the special geometry of the cobweb. By modifying the method slightly to take care of the cobweb geometry, Izmailian, Kennna, and Wu succeeded in establishing the Tan-Zhou-Yang conjecture using a modified Laplacian approach [7].

In this paper we consider another special geometry of a network, a globe, or a cobweb with its boundary collapsed into one node resulting in a network in the shape of a globe shown in the right panel of Fig. 1. Thus, an $m\times n$ cobweb network of m rows and n columns becomes a globe with $m-1$ latitudes and n longitudes. The example of $m=6, n=12$ is shown in Fig. 1; however, due to its special geometry, both the Laplacian and the Izmailian-Kenna-Wu modified Laplacian approaches are difficult to apply and an alternative consideration is needed.

Studies of the resistance problem have also been carried out independently by Tan *et al.* along a route that we shall refer to as the method of direct evaluation [6,8–10]. The direct method is useful in cases when there exists a special node such as a pole of the globe or the center of the cobweb, with all other nodes connected to it equally along longitudes of a globe or the radii of a cobweb. This unique connectivity makes it possible to compute the potential between two nodes by computing separately their relative potentials with respect to the special node and take the difference. One thus circumvents the need of diagonalizing a nonregular Laplacian matrix. The direct method of computing resistances had been applied successively to the cobweb network for fixed values of m up to $m=4$ [6,8,9]. It has also been used recently to compute the resistances in a fan network [11]. In this paper we apply the direct method to solve the globe problem.

II. EQUIVALENT RESISTANCE: THE MAIN RESULT

We consider a globe with n longitudes and $m-1$ latitudes shown in Fig. 1. Bonds in the longitude and latitude directions have respective resistances r_0 and r and we let the south pole O be the origin of coordinates. We define a variable L_i and

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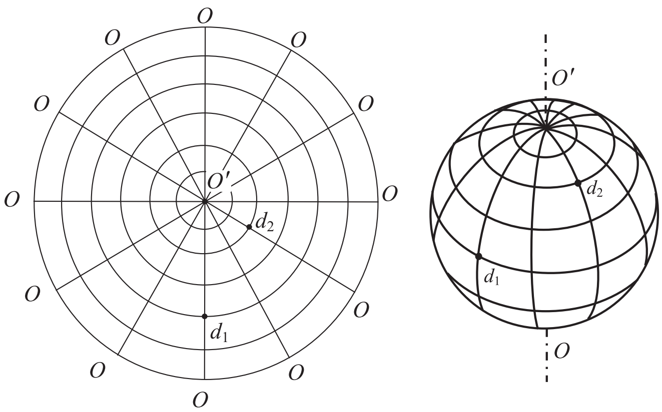

FIG. 1. A 6×12 cobweb network with its boundary collapsed into a single node O, resulting in a globe with 5 latitudes and 12 longitudes. Bonds in longitude and latitude directions represent resistors r_0 and r, respectively. The cobweb center O' is the north pole and the cobweb boundary collapses into the south pole O.

for later use λ_i and $\bar{\lambda}_i$ by

$$\begin{aligned} \lambda_i \equiv e^{2L_i} &= 1 + h - h\cos\theta_i + \sqrt{(1+h-h\cos\theta_i)^2 - 1}, \\ \bar{\lambda}_i \equiv e^{-2L_i} &= 1 + h - h\cos\theta_i - \sqrt{(1+h-h\cos\theta_i)^2 - 1}, \\ \cosh 2L_i &= 1 + h - h\cos\theta_i, \end{aligned} \tag{1}$$

where

$$h = r/r_0, \quad \theta_i = (i-1)\pi/m, \quad i = 1,2,\ldots,m.$$

We denote nodes of the network by the coordinate $\{x,y\}$, where $x = 1,2,\ldots,n$ and $y = 0,1,2,\ldots,m$, with $y = 1$ denoting the latitude just above the pole O and $x = 1$ any longitudinal under the cyclic boundary condition. We find the resistance between two nodes $d_1 = \{1,y_1\}$ and $d_2 = \{x+1,y_2\}$ to be given by the expression

$$R^{\text{globe}}_{m\times n}(\{1,y_1\},\{x+1,y_2\}) = \frac{(y_1-y_2)^2}{mn} r_0 + \frac{r}{m}\sum_{i=2}^{m} \frac{\cosh(nL_i)(\sin^2 y_1\theta_i + \sin^2 y_2\theta_i) - 2\cosh[(n-2x)L_i]\sin(y_1\theta_i)\sin(y_2\theta_i)}{\sinh(2L_i)\sinh(nL_i)}. \tag{2}$$

In particular, we have the following special cases.

Case 1. When d_1 and d_2 are on the same longitude at $\{1,y_1\}$ and $\{1,y_2\}$, we have

$$\begin{aligned} R^{\text{long}}_{m\times n}(d_1,d_2) &= \frac{(y_1-y_2)^2}{mn} r_0 \\ &\quad + \frac{r}{m}\sum_{i=2}^{m}(\sin y_1\theta_i - \sin y_2\theta_i)^2\left(\frac{\coth(nL_i)}{\sinh(2L_i)}\right). \end{aligned} \tag{3}$$

Case 2. When d_1 and d_2 are on the same latitude at $\{1,y\}$ and $\{x+1,y\}$, we have

$$R^{\text{latt}}_{m\times n}(d_1,d_2) = \frac{4r}{m}\sum_{i=2}^{m}\frac{\sinh(xL_i)\sinh[(n-x)L_i]}{\sinh(2L_i)\sinh(nL_i)}[\sin^2(y\theta_i)], \tag{4}$$

The expression (4) is invariant under $x \leftrightarrow (n-x)$ as expected.

Case 3. The resistance between a node at $\{x,y\}$ and the north pole O' is

$$\begin{aligned} R_{m\times n}(\{x,y\},O') &= \frac{(m-y)^2}{mn} r_0 + \frac{r}{m}\sum_{i=2}^{m}\sin^2(y\theta_i)\left(\frac{\coth(nL_i)}{\sinh(2L_i)}\right). \end{aligned} \tag{5}$$

Case 4. The resistance between the two poles O and O' is

$$R_{m\times n}(O,O') = mr_0/n. \tag{6}$$

III. DERIVATION OF THE MAIN RESULT (2)

A. Expressing the resistance in terms of longitudinal currents

To compute the resistance between two nodes $d_1 = \{1,y_1\}$ and $d_2 = \{x+1,y_2\}$ we inject a current J into the network at d_1 and exit the current at d_2. We denote the currents in all segments of the network as shown in Fig. 2. Then by Ohm's law the potential differences between d_1, d_2, and the north pole O' are, respectively,

$$U^{\text{globe}}_{m\times n}(d_1,O') = r_0\sum_{i=y_1+1}^{m} I_1^{(i)},$$

$$U^{\text{globe}}_{m\times n}(O',d_2) = -r_0\sum_{i=y_2+1}^{m} I_{x+1}^{(i)},$$

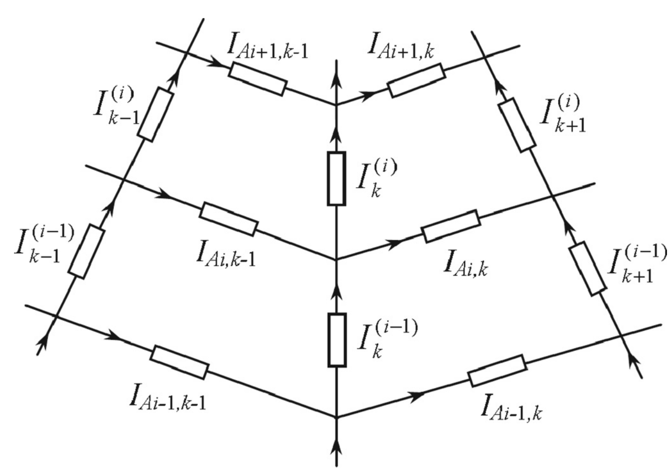

FIG. 2. Segment of the globe with current directions.

where $I_1^{(i)}$ denotes currents along the longitude 1 and $I_{x+1}^{(i)}$ denotes currents along the longitudinal $x+1$. It then follows from Ohm's law that the resistance between d_1 and d_2 is

$$R_{m\times n}^{\rm globe}(\{1,y_1\},\{x+1,y_2\}) = \frac{r_0}{J}\left(\sum_{i=y_1+1}^{m} I_1^{(i)} - \sum_{i=y_2+1}^{m} I_{x+1}^{(i)}\right). \tag{7}$$

Therefore, we need to find the longitudinal currents $I_1^{(i)}$ and $I_{x+1}^{(i)}$. This is the main objective of this paper.

B. Matrix equation for longitudinal currents

Analysis of the longitudinal currents is best carried out in terms of a matrix equation. Early discussions along this line are due to Tan *et al.* [6,8–10]. A similar analysis for a fan network has been given recently in [11].

A segment of the globe network is shown in Fig. 2 with current labeling and we focus on the two upper rectangular meshes. Around the two meshes there are five longitudinal currents $I_{k-1}^{(i)},I_k^{(i)},I_{k+1}^{(i)},I_k^{(i-1)},I_k^{(i+1)}$ and four horizontal currents $I_{Ai,k}$. The potential across each current segment is either $I_k^{(i)}r_0$ or $I_{Ai,k}r$. The Kirchhoff law says that the sum of the potentials around any closed loop is equal to zero. Applying this to the outer perimeter of the two meshes gives an equation relating the four horizontal currents. Furthermore, the sum of all currents at a node must be zero. Applying this Kirchhoff rule to the two upper consecutive nodes on the longitude k, one obtains two more equations relating the four horizontal currents. However, it can be seen from Fig. 2 that the four horizontal currents enter all three equations only in the combination of $\Im_1 = I_{Ai+1,k-1} - I_{Ai+1,k}$ and $\Im_2 = I_{Ai,k-1} - I_{Ai,k}$. Thus one can eliminate $\Im_1$ and $\Im_2$ from the three equations. This gives the relation

$$I_{k+1}^{(i)} = -I_{k-1}^{(i)} + 2(1+h)I_k^{(i)} - hI_k^{(i+1)} - hI_k^{(i-1)} \tag{8}$$

connecting the five longitudinal currents. After taking into account modifications at $i=1,m$ [11], (8) can be written in a matrix form

$$\mathbf{I}_{k+1} = \mathbf{A}_m\mathbf{I}_k - \mathbf{I}_{k-1}, \tag{9}$$

where $\mathbf{A}_m$ and $\mathbf{I}_k$ are

$$\mathbf{A}_m = \begin{pmatrix} 2+h & -h & 0 & 0 & \cdots & 0 \\ -h & 2(1+h) & -h & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -h & 2(1+h) & -h \\ 0 & \cdots & 0 & 0 & -h & 2+h \end{pmatrix}, \quad \mathbf{I}_k = \begin{pmatrix} I_k^{(1)} \\ I_k^{(2)} \\ \vdots \\ I_k^{(m-1)} \\ I_k^{(m)} \end{pmatrix}. \tag{10}$$

It is understood that we have the cyclic condition

$$\mathbf{I}_0 = \mathbf{I}_n, \quad \mathbf{I}_{n+1} = \mathbf{I}_1. \tag{11}$$

We consider the solution of (9) in the next section.

C. General solution of the matrix equation

In this section we consider the solution of (9) in the absence of an injected current, namely, $J=0$. The eigenvalues t_i, $i=1,2,\ldots,m$, of $\mathbf{A}_m$ are the m solutions of the equation

$$\det|\mathbf{A}_m - t\bar{\mathbf{I}}_m| = 0, \tag{12}$$

where $\bar{\mathbf{I}}_m$ is the $m\times m$ identity matrix. Since $\mathbf{A}_m$ is Hermitian it can be diagonalized by a similarity transformation to yield

$$\mathbf{P}_m\mathbf{A}_m(\mathbf{P}_m)^{-1} = \mathbf{\Lambda}_m, \tag{13}$$

where $\mathbf{\Lambda}_m$ is a diagonal matrix with eigenvalues t_i of $\mathbf{A}_m$ in the diagonal and column vectors of $(\mathbf{P}_m)^{-1}$ are eigenvectors of $\mathbf{A}_m$.

It can be verified that we have

$$\mathbf{P}_m = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & \cdots & 1/\sqrt{2} \\ \cos\left(1-\frac{1}{2}\right)\theta_2 & \cos\left(2-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left(1-\frac{1}{2}\right)\theta_m & \cos\left(2-\frac{1}{2}\right)\theta_m & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_m \end{pmatrix}, \tag{14}$$

$$(\mathbf{P}_m)^{-1} = \frac{2}{m}\begin{pmatrix} 1/\sqrt{2} & \cos\left(1-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(1-\frac{1}{2}\right)\theta_m \\ 1/\sqrt{2} & \cos\left(2-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(2-\frac{1}{2}\right)\theta_m \\ \vdots & \vdots & \ddots & \vdots \\ 1/\sqrt{2} & \cos\left(m-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_m \end{pmatrix}, \tag{15}$$

where $\theta_i = (i-1)\pi/m$, and

$$t_i = 2(1+h) - 2h\cos\theta_i = \lambda_i + \bar{\lambda}_i = 2\cosh(2L_i), \quad i = 1,2,3,\ldots,m, \tag{16}$$

where we have made use of (1). We apply $\mathbf{P}_m$ on the left-hand side of (9) and write

$$\mathbf{X}_k \equiv \mathbf{P}_m \mathbf{I}_k \quad \text{or} \quad \mathbf{I}_k = (\mathbf{P}_m)^{-1}\mathbf{X}_k. \tag{17}$$

After making use of (13), we obtain the equation

$$\mathbf{X}_{k+1} = \mathbf{\Lambda}_m \mathbf{X}_k - \mathbf{X}_{k-1}. \tag{18}$$

We let the ith element of the column vector $\mathbf{X}_k$ be $X_k^{(i)}$. Then (18) gives

$$X_{k+1}^{(i)} = t_i X_k^{(i)} - X_{k-1}^{(i)}, \quad i = 1,2,\ldots,m, \tag{19}$$

which is a set of recurrence relations for $X_k^{(i)}$.

For $i = 1$, the solution of (19), which we will make use of later, is particularly simple. Since $\theta_1 = 0$ and $L_1 = 0$, we have $t_1 = 2$. Then (19) becomes

$$X_{k+1}^{(1)} = 2X_k^{(1)} - X_{k-1}^{(1)}, \quad k = 1,2,\ldots,n-1, \tag{20}$$

which together with the cyclic condition $X_0^{(1)} = X_n^{(1)}$ is a set of $n-1$ linear relations for n unknowns $X_k^{(1)}, k = 1,2,\ldots,n$, which is insufficient. However, other than the trivial solution $X_k^{(1)} = 0$, which is useless, we have also the obvious solution that all $X_k^{(1)}$ are equal, namely,

$$X_1^{(1)} = X_2^{(1)} = \cdots = X_n^{(1)}. \tag{21}$$

For $i > 1$, the recurrence relation (19) can be solved by the method of generating function. We define generating function

$$G(s) = \sum_{k=1}^{\infty} X_k^{(i)} s^k. \tag{22}$$

Multiplying (19) by s^k and summing both sides of the equation from $k = 1$ to $k = \infty$ yields

$$\frac{1}{s}\left[G(s) - X_1^{(i)}s - X_2^{(i)}s^2\right] = t_i\left[G(s) - X_1^{(i)}s\right] - sG(s),$$

from which we solve for $G(s)$, obtaining

$$G(s) = \frac{X_1^{(i)}s + \left(X_2^{(i)} - t_i X_1^{(i)}\right)s^2}{1 - t_i s + s^2}. \tag{23}$$

The partial fraction expansion of (23) by using $1 - t_i s + s^2 = (1-\lambda_i s)(1-\bar{\lambda}_i s)$, where λ_i and $\bar{\lambda}_i$ are defined in (1), gives

$$\frac{1}{1 - t_i s + s^2} = \frac{1}{\lambda_i - \bar{\lambda}_i}\left(\frac{\lambda_i}{1-\lambda_i s} - \frac{\bar{\lambda}_i}{1-\bar{\lambda}_i s}\right),$$

which we substitute into (23). By expanding the right-hand side of (23) into a series in s by making use of $(1-z)^{-1} = 1 + z + z^2 + \cdots$ and comparing both sides term by term, we obtain, after making use of the identity $F_k^{(i)} - t_i F_{k-1}^{(i)} = -F_{k-2}^{(i)}$, the solution of $X_k^{(i)}$ in terms of a given initial condition of $X_1^{(i)}$ and $X_2^{(i)}$,

$$X_k^{(i)} = X_2^{(i)} F_{k-1}^{(i)} - X_1^{(i)} F_{k-2}^{(i)}, \quad i > 1,\ k \geqslant 1, \tag{24}$$

where

$$F_k^{(i)} = \frac{\lambda_i^k - \bar{\lambda}_i^k}{\lambda_i - \bar{\lambda}_i} = \frac{\sinh(2kL_i)}{\sinh(2L_i)}. \tag{25}$$

In a similar fashion, by considering the generating function (22) with a summation over k from $k = u+1$ to ∞ with a given initial condition of $X_{u+2}^{(i)}$ and $X_{u+1}^{(i)}$, where $u \geqslant 0$ is arbitrary, we obtain the solution

$$X_k^{(i)} = X_{u+2}^{(i)} F_{k-u-1}^{(i)} - X_{u+1}^{(i)} F_{k-u-2}^{(i)}, \quad i > 1,\quad u \geqslant 0,\quad k \geqslant u+1. \tag{26}$$

Note that (26) reduces to (24) when $u = 0$.

D. Boundary conditions with input and output currents

While either (24) or (26) serves to determine $\mathbf{I}_k$ when there is no external current injected to the network, to compute the resistance between nodes $d_1 = d_1(1,y_1)$ and $d_2 = d_2(x+1,y_2)$ we need to inject current J at d_1 and exit the current at d_2. Then (24) holds only for $1 \leqslant k \leqslant x+1$. For k in the range of $x+1 \leqslant k \leqslant n+1$, however, we need to use (26) with $u = x$. Thus the injection of J at $d_1(1,y_1)$ and the exit of J at $d_2 = d_2(x+1,y_2)$ specialize (9) for $k = 1$ and $k = x+1$ to

$$\mathbf{I}_2 = \mathbf{A}_m \mathbf{I}_1 - \mathbf{I}_n - J\mathbf{H}_1, \tag{27}$$

$$\mathbf{I}_{x+2} = \mathbf{A}_m \mathbf{I}_{x+1} - \mathbf{I}_x - J\mathbf{H}_2, \tag{28}$$

where we have made use of the cyclic condition $\mathbf{I}_0 = \mathbf{I}_n$ and $\mathbf{H}_1$ and $\mathbf{H}_2$ are column matrices with elements

$$(H_1)_i = h\left(-\delta_{i,y_1} + \delta_{i,y_1+1}\right),$$
$$(H_2)_i = h\left(\delta_{i,y_2} - \delta_{i,y_2+1}\right)$$

or, equivalently,

$$\mathbf{H}_1 = [\overbrace{0,\ldots,0,-h,h}^{\text{from 0th to } (y_1+1)\text{th}},0,\ldots,0\,]^T,$$
$$\mathbf{H}_2 = [\overbrace{0,\ldots,0,h,-h}^{\text{from 0th to } (y_2+1)\text{th}},0,\ldots,0]^T,$$

where $[\,]^T$ denote matrix transposes. Applying $\mathbf{P}_m$ to (27) and (28) on the left-hand sides, we are led to

$$\mathbf{X}_2 = \mathbf{\Lambda}_m \mathbf{X}_1 - \mathbf{X}_n - hJ\mathbf{D}_1, \tag{29}$$

$$\mathbf{X}_{x+2} = \mathbf{\Lambda}_m \mathbf{X}_{x+1} - \mathbf{X}_x - hJ\mathbf{D}_2, \tag{30}$$

where $h\mathbf{D}_1 = \mathbf{P}_m\mathbf{H}_1$ and $h\mathbf{D}_2 = \mathbf{P}_m\mathbf{H}_2$ or, equivalently,

$$\mathbf{D}_1 = [\zeta_{1,1},\zeta_{1,2},\ldots,\zeta_{1,i},\ldots,\zeta_{1,m-1},\zeta_{1,m}]^T,$$
$$\zeta_{1,i} = P_{y_1,i} - P_{y_1+1,i} = -\cos\left(y_1 - \tfrac{1}{2}\right)\theta_i + \cos\left(y_1 + \tfrac{1}{2}\right)\theta_i = -2\sin(y_1\theta_i)\sin(\theta_i/2), \tag{31}$$

$$\mathbf{D}_2 = [\zeta_{2,1},\zeta_{2,2},\ldots,\zeta_{2,i},\ldots,\zeta_{2,m-1},\zeta_{2,m}]^T,$$
$$\zeta_{2,i} = P_{y_2,i} - P_{y_2+1,i} = \cos\left(y_2 - \tfrac{1}{2}\right)\theta_i - \cos\left(y_2 + \tfrac{1}{2}\right)\theta_i = 2\sin(y_2\theta_i)\sin(\theta_i/2). \tag{32}$$

Explicitly, (29) and (30) read

$$X_2^{(i)} = t_i X_1^{(i)} - X_n^{(i)} - hJ\zeta_{1,i}, \tag{33}$$

$$X_{x+2}^{(i)} = t_i X_{x+1}^{(i)} - X_x^{(i)} - hJ\zeta_{2,i}, \tag{34}$$

where $t_i = 2\cosh 2L_i$.

To determine $X_1^{(i)}, X_{x+1}^{(i)}$ needed in our resistance calculation (7), we set $k = x, x+1$ in (24), $u = x$ and $k = n, n+1$ in (26), and make use of the cyclic condition (11) $X_{n+1}^{(i)} = X_1^{(i)}$. Together with (33) and (34) this gives six equations relating the six unknowns $X_1^{(i)}, X_2^{(i)}, X_n^{(i)}, X_x^{(i)}, X_{x+1}^{(i)}, X_{x+2}^{(i)}$,

$$\begin{pmatrix} F_{x-2}^{(i)} & -F_{x-1}^{(i)} & 0 & 1 & 0 & 0 \\ F_{x-1}^{(i)} & -F_x^{(i)} & 0 & 0 & 1 & 0 \\ t_i & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & F_{n-x-2}^{(i)} & -F_{n-x-1}^{(i)} \\ 1 & 0 & 0 & 0 & F_{n-x-1}^{(i)} & -F_{n-x}^{(i)} \\ 0 & 0 & 0 & -1 & t_i & -1 \end{pmatrix} \begin{pmatrix} X_1^{(i)} \\ X_2^{(i)} \\ X_n^{(i)} \\ X_x^{(i)} \\ X_{x+1}^{(i)} \\ X_{x+2}^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ hJ\zeta_{1,i} \\ 0 \\ 0 \\ hJ\zeta_{2,i} \end{pmatrix}, \quad i > 1, \tag{35}$$

where $t_i = 2\cosh(2L_i)$ and $F_k^{(i)} = \sinh(2kL_i)/\sinh(2L_i)$. Solving (35), we obtain after some algebra and reduction the two solutions needed in our resistance calculation (7),

$$\begin{aligned} X_1^{(i)} &= \frac{\left(F_{n-x}^{(i)} + F_x^{(i)}\right)\zeta_{2,i} + F_n^{(i)}\zeta_{1,i}}{4\sinh^2 nL_i} hJ \\ &= hJ\left(\frac{\left(F_{n-x}^{(i)} + F_x^{(i)}\right)\sin(y_2\theta_i) - F_n^{(i)}\sin(y_1\theta_i)}{2\sinh^2 nL_i}\right)\sin(\theta_i/2), \quad i > 1, \end{aligned} \tag{36}$$

$$\begin{aligned} X_{x+1}^{(i)} &= \frac{\left(F_{n-x}^{(i)} + F_x^{(i)}\right)\zeta_{1,i} + F_n^{(i)}\zeta_{2,i}}{4\sinh^2 nL_i} hJ \\ &= hJ\left(\frac{-\left(F_{n-x}^{(i)} + F_x^{(i)}\right)\sin(y_1\theta_i) + F_n^{(i)}\sin(y_2\theta_i)}{2\sinh^2 nL_i}\right)\sin(\theta_i/2), \quad i > 1. \end{aligned} \tag{37}$$

Solutions (36) and (37) are useful for $i > 1$. For $i = 1$ (36) and (37) give the trivial solutions $X_1^{(1)} = X_{x+1}^{(1)} = 0$; however, when $i = 1$ we have $\zeta_{1,i} = \zeta_{2,i} = 0$ so (33) and (34) reduce to (20). Then using the same argument leading to (21), we again obtain $X_1^{(1)} = X_2^{(1)} = \cdots = X_n^{(1)}$. This permits us to write

$$X_1^{(1)} = \frac{1}{n}\sum_{k=1}^n X_k^{(1)} = \frac{1}{n}\sum_{k=1}^n\sum_{j=1}^m \left[(\mathbf{P}_m)_{1j} I_k^{(j)}\right] = \frac{1}{\sqrt{2}\,n}\sum_{i=1}^m\sum_{k=1}^n I_k^{(i)}, \tag{38}$$

where we have made use of $(\mathbf{P}_m)_{1j} = 1/\sqrt{2}$.

The summations in (38) are taken over all longitudinal current segments on the globe. Since the current J flows from a node at latitude y_1 to a node at latitude y_2, by conservation of current the summation over segments at a given latitude i must yield J for $y_1 < i \leqslant y_2$ and zero otherwise, namely,

$$\begin{aligned} \sum_{k=1}^n I_k^{(i)} &= J \quad \text{for} \quad y_1 < i < y_2 + 1 \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{39}$$

so (38) gives the simple result

$$X_1^{(1)} = \frac{J}{\sqrt{2}\,n}(y_2 - y_1). \tag{40}$$

E. Equivalent resistance

We are now in a position to evaluate the resistance (7). From (17) we have

$$I_1^{(i)} = \sum_{j=1}^m [(\mathbf{P}_m)^{-1}]_{ij} X_1^{(j)}.$$

Using $(\mathbf{P}_m)^{-1}$ given by (15) with $[(\mathbf{P}_m)^{-1}]_{i1} = \sqrt{2}/m$ for all i, it is clear that the $j = 1$ term in the summation needs to be singled out. This gives

$$I_1^{(i)} = \frac{\sqrt{2}}{m} X_1^{(1)} + \frac{2}{m}\sum_{j=2}^m X_1^{(j)} \cos\left(i - \frac{1}{2}\right)\theta_j \tag{41}$$

and thus

$$\sum_{i=y_1+1}^m I_1^{(i)} = \frac{\sqrt{2}}{m}(m - y_1)X_1^{(1)} - \frac{1}{m}\sum_{j=2}^m X_1^{(j)}\left(\frac{\sin(y_1\theta_j)}{\sin\left(\frac{1}{2}\theta_j\right)}\right), \tag{42}$$

where we have used the formula

$$\sum_{i=y+1}^{m} \cos\left(i-\frac{1}{2}\right)\theta_j = -\left(\frac{\sin(y\theta_j)}{2\sin\left(\frac{1}{2}\theta_j\right)}\right), \tag{43}$$

which can be established by using the identity $\sum_{k=1}^{n}\cos(k-\frac{1}{2})x = \sin(nx)/2\sin(x/2)$ [12].

Substituting (40) into (42), we obtain

$$\sum_{i=y_1+1}^{m} I_1^{(i)} = \frac{J}{mn}(m-y_1)(y_2-y_1) - \frac{1}{m}\sum_{j=2}^{m} X_1^{(j)}\left(\frac{\sin(y_1\theta_j)}{\sin\left(\frac{1}{2}\theta_j\right)}\right). \tag{44}$$

Similarly, we also obtain

$$\sum_{i=y_2+1}^{m} I_{x+1}^{(i)} = \frac{J}{mn}(m-y_2)(y_2-y_1) - \frac{1}{m}\sum_{j=2}^{m} X_{x+1}^{(j)}\left(\frac{\sin(y_2\theta_j)}{\sin\left(\frac{1}{2}\theta_j\right)}\right). \tag{45}$$

Substituting (44) and (45) into (7), we obtain

$$R_{m\times n}^{\text{globe}}(d_1,d_2) = \frac{r_0}{m}\left(\frac{(y_2-y_1)^2}{n} + \frac{1}{J}\sum_{i=2}^{m}\frac{X_{x+1}^{(i)}\sin(y_2\theta_i) - X_1^{(i)}\sin(y_1\theta_i)}{\sin\left(\frac{1}{2}\theta_i\right)}\right). \tag{46}$$

Finally, we obtain our main result (2) by further substituting $X_1^{(i)}$ and $X_{x+1}^{(i)}$ from (36) and (37) into (46).

F. Special cases

The special cases can be summarized as follows.

Case 1. When $d_1 = \{1,y_1\}$ and $d_2 = \{1,y_2\}$ are on the same longitude, we take $x = 0$ and (2) reduces immediately to (3).

Case 2. When $d_1 = \{1,y\}$ and $d_2 = \{x+1,y\}$ are on the same latitude y, (2) immediately reduces to (4).

Case 3. The resistance between a node at $\{x,y\}$ and the north pole O' is obtained by setting $y_1 = y$ and $y_2 = m$ in (3). This gives (5).

Case 4. The resistance between the two poles is obtained by setting $y_1 = 0$ and $y_2 = m$ in (3). This gives $R_{m\times n}(O,O') = mr_0/n$. This result can also be deduced by considering $R_{m\times n}(O,O')$ as connecting n linear chains of resistance mr_0 each in parallel, since by symmetry there are no currents in the horizontal direction.

IV. SUMMARY AND DISCUSSION

Wu [4] established a theorem that computes the equivalent resistance between two nodes in a resistor network using the Laplacian approach. For the $m \times n$ network the results are in the form of a double summation. Additional work is required to reduce this to a single summation.

An alternative direct approach of computing resistances had been developed by Tan *et al.* [6,8–10] that, when applied to the cobweb and globe networks, gives the result in terms of a single summation. This offers a direct and somewhat simpler approach. The direct method has been used by the present authors [11] to obtain the two-point resistance in a fan network. Here we have used the direct method to compute resistances in a globe network equivalent to a cobweb with the boundary collapsed into one point. Our main result is (2), which gives the resistance between any two nodes of the globe. Various special cases of the main result were presented.

It is useful to summarize the main idea of the direct approach, which is a simple application of Ohm's law. To compute the resistance R_{12} between two nodes 1 and 2, one injects a current I into the network at 1 with the current exiting at 2 and computes the potential differences $U_1 = U_{1O'}$ and $U_2 = U_{2O'}$ between 1 and 2 and the pole O'. Then the resistance is $R_{12} = |U_1 - U_2|/I$.

It is also instructive to comment on why the Laplacian method cannot be used. While it is tempting to apply the Laplacian method by considering the globe as a cobweb with zero resistances along its boundary, since elements of the Laplacian are conductances that are infinite for zero resistances, the application is not easily done. It is simpler and far easier to use the direct approach.

The direct method can be extended to impedance networks since Ohm's law based on which the method is formulated is applicable to impedances. This is more advantageous than the Laplacian method, which needs to be modified when dealing with impedance networks as the Laplacian matrix is generally complex and non-Hermitian, which require special considerations [5].

Finally, we remark that at large distances the two-point resistance is expected to diverge logarithmically. This is what had been found in regular lattices [13,14] and in the cobweb [15]. The modification of the network by one or two special nodes should not alter its macroscopic behavior.

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